Evidence for power-law anti-correlations in complex networks

Diego Rybski
Hernán D. Rozenfeld, Jürgen P. Kropp

SOE 18.6
DPG Frühjahrstagung 2010
Regensburg; H44
25.3.2010 – 11:45

PIK
Potsdam Institute for Climate Impact Research

Motivation

Networks consist of simple elements but can comprise complex properties

- broad degree distribution
- clustering
- modularity
- ...

Motivation

Degree correlations

- likelihood that nodes of given degree are connected
- assortative/disassortative mixing:
  positive/negative correlations
- measures:
  - Pearson correlation coefficient
  - average nearest neighbor degree
  - conditional probability $p(k_1, k_2)$
  - ...

disassortativity tightly related to *fractality* of complex networks

Average nearest neighbor degree

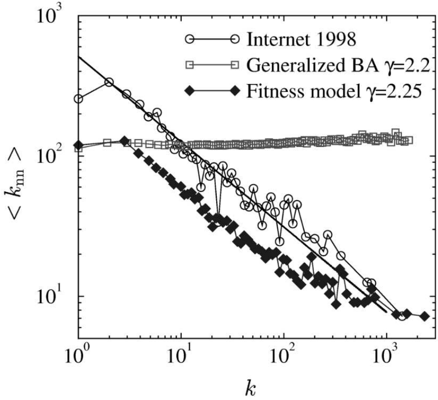


FIG. 2. The average connectivity $\langle k_{nn} \rangle$ of the nearest neighbors of a node depending on its connectivity k for the 1998 snapshot of the Internet, the generalized BA model with $\gamma = 2.2$ (Ref. [8]), and the fitness model (Ref. [18]). The solid line has a slope -0.5. The scattered results for very large k are due to statistical fluctuations.

Figure: R. Pastor-Satorras et al. PRL 2001.

Conditional probability $p(k_1, k_2)$

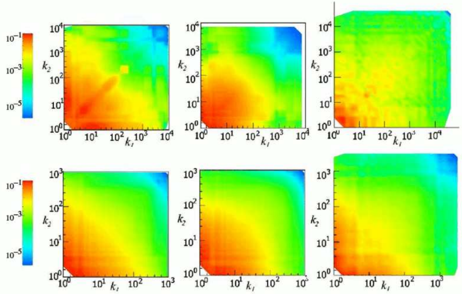


FIG. 1 (color online). The joint degree distribution $P(k_1, k_2)$ of WWW (top row) and Internet at the router level (bottom row) before renormalization (left), after renormalization forbidding multiple links (center), and including multiple links (right).

Figure: L.K. Gallos et al. PRL 2008.

Motivation

But:

- only correlations between nearest neighbor nodes, i.e. distance 1
- much of the rich topological information gets lost
- **how can correlations be measured at larger distances?**

Fluctuation Analysis

We propose a Degree Fluctuation Analysis (kFA):

Fluctuation Analysis

We propose a Degree Fluctuation Analysis (kFA):

1. Consider the shortest path between the nodes i and j (if it is not unique, we consider an arbitrary one).

Fluctuation Analysis

We propose a Degree Fluctuation Analysis (kFA):

1. Consider the shortest path between the nodes i and j (if it is not unique, we consider an arbitrary one).
2. Extract the sequence of degrees (k_l) along this path of length d_{ij}.

Fluctuation Analysis

We propose a Degree Fluctuation Analysis (kFA):

1. Consider the shortest path between the nodes i and j (if it is not unique, we consider an arbitrary one).
2. Extract the sequence of degrees (k_l) along this path of length d_{ij}.
3. Calculate the average $K_{ij} = \langle k_l \rangle$ of the sequence (k_l).

Fluctuation Analysis

We propose a Degree Fluctuation Analysis (kFA):

1. Consider the shortest path between the nodes i and j (if it is not unique, we consider an arbitrary one).
2. Extract the sequence of degrees (k_l) along this path of length d_{ij}.
3. Calculate the average $K_{ij} = \langle k_l \rangle$ of the sequence (k_l).
4. Find the shortest paths between all pairs of nodes and determine the corresponding averages, $K_{ij} \ \forall \ i \neq j$.

Fluctuation Analysis

We propose a Degree Fluctuation Analysis (kFA):

1. Consider the shortest path between the nodes i and j (if it is not unique, we consider an arbitrary one).
2. Extract the sequence of degrees (k_l) along this path of length d_{ij}.
3. Calculate the average $K_{ij} = \langle k_l \rangle$ of the sequence (k_l).
4. Find the shortest paths between all pairs of nodes and determine the corresponding averages, $K_{ij} \ \forall \ i \neq j$.
5. Calculate the fluctuation function $F(d) = \sigma(K_{ij}|d)$, the conditional standard deviation of the K_{ij} at distance d.

In analogy to Fluctuation Analysis in time series analysis

Long-range (anti-) correlations

If the covariance, $C(d) \sim \langle (k_i - \langle k \rangle)(k_j - \langle k \rangle) | d \rangle$, between the degrees at distance d scales as

$C(d) \sim d^{-\gamma}$	for positive correlations (assortative) or
$C(d) \sim -(d^{-\gamma})$	for negative correlations (disassortative)

then we expect

$$F(d) \sim d^{\alpha_k},$$

where $\alpha_k = -\gamma/2$.

Fluctuation exponent differs by 1 from usual Hurst-like exponent: $\alpha = \alpha_k + 1$.

Long-range (anti-) correlations

If the covariance, $C(d) \sim \langle (k_i - \langle k \rangle)(k_j - \langle k \rangle) | d \rangle$, between the degrees at distance d scales as

$C(d) \sim d^{-\gamma}$	for positive correlations (assortative) or
$C(d) \sim -(d^{-\gamma})$	for negative correlations (disassortative)

then we expect

$$F(d) \sim d^{\alpha_k},$$

where $\alpha_k = -\gamma/2$.

Fluctuation exponent differs by 1 from usual Hurst-like exponent: $\alpha = \alpha_k + 1$.

$\alpha_k > -1/2$	positive correlations (assortative)
$\alpha_k = -1/2$	uncorrelated
$\alpha_k < -1/2$	negative correlations (disassortative)

Long-range (anti-) correlations

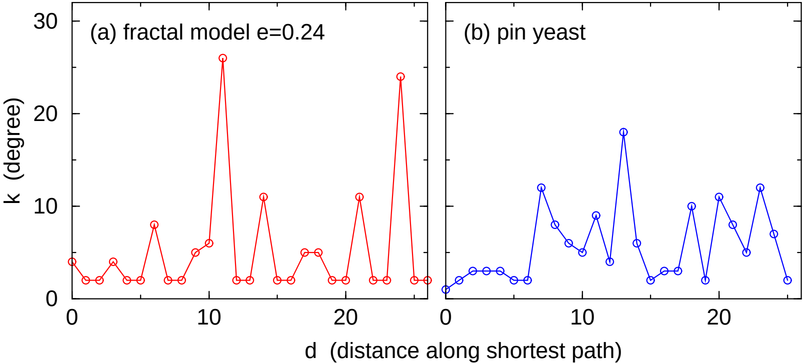


Figure: Degrees of nodes versus distance along shortest paths for (a) fractal network model and (b) pin yeast network.

1. Barabási-Albert model

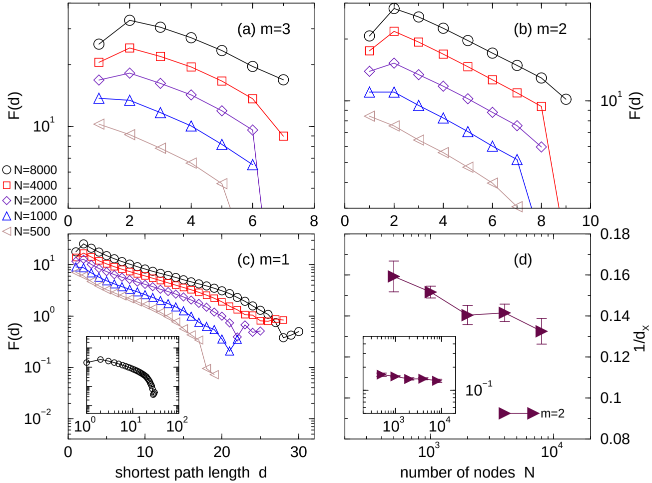


Figure: Degree fluctuation functions for the BA model. (a-c) show $F(d)$, (d) shows for $m = 2$ the slopes of exponential fits as a function of the network sizes. 100 configurations.

2. Cayley tree at percolation transition

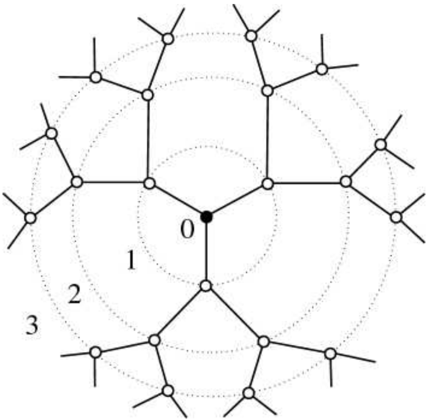


Figure: $z = 3$ (source: wikipedia)

percolation transition: $p_{\mathrm{c}} = \frac{1}{z-1}$

topological dimension of giant component: $d_{\mathrm{f}} = 2$

2. Cayley tree at percolation transition

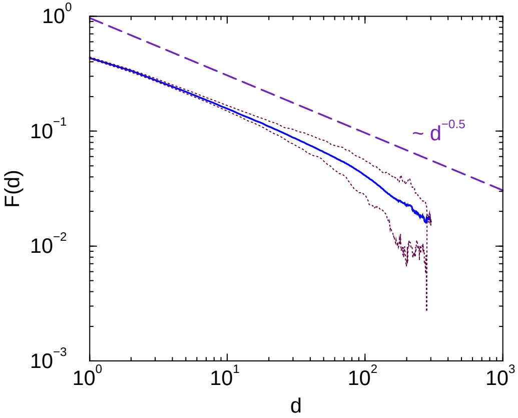


Figure: Degree fluctuation function of the Cayley tree at percolation transition ($z = 3$, $n = 150$). Dotted maroon lines: quantiles enclosing 90% (100 configurations).

3. Fractal network model

C.M. Song, S. Havlin, and H.A. Makse, nature physics, 2006.

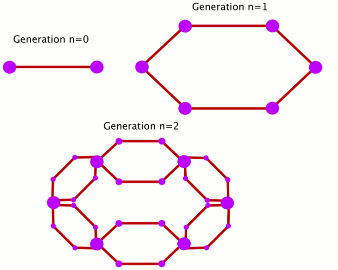


Fig. 1. Construction of a pure fractal network. Example of network model with parameters $n = 0, 1, 2; m = 2; x = 2; e = 0$.

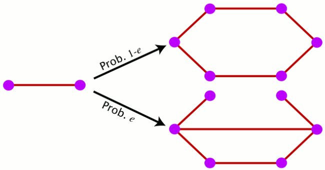


Fig. 2. Construction of network. With probability e the link between hub remains, otherwise, with probability $1 - e$ it is replaced for another link between new nodes.

Figure: generation *n*, *m* new nodes, *x* new links, probability *e* (source: H.D. Rozenfeld and H.A. Makse, 2009)

fractal dimension: $$d_f = \frac{\ln(2m+x)}{\ln(3-2e)}$$

degree distribution: $$p(k) \sim k^{-\left(1+\frac{\ln(2m+x)}{\ln m}\right)}$$

3. Fractal network model

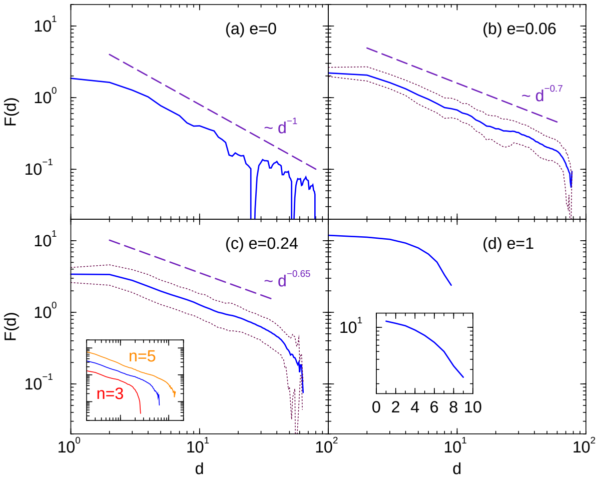


Figure: Degree fluctuation functions for the fractal model ($n = 4$). Dotted maroon lines: quantiles enclosing 90% (250 configurations). Inset in (c): $n = 3$ and $n = 5$ (25 configurations).

3. Fractal network model

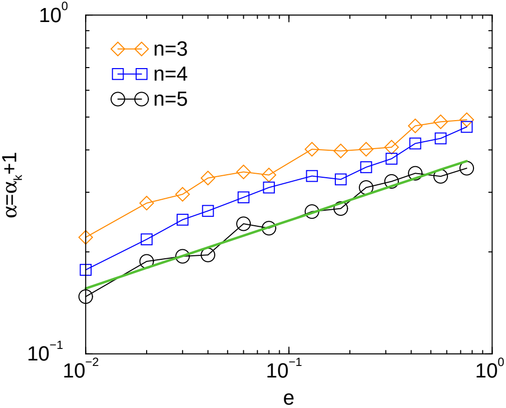


Figure: Exponents $\alpha = \alpha_k + 1$ of the fractal network model. Power-law fit for $n = 5$: $\alpha \sim e^{\epsilon}$, $\epsilon \approx 0.2$.

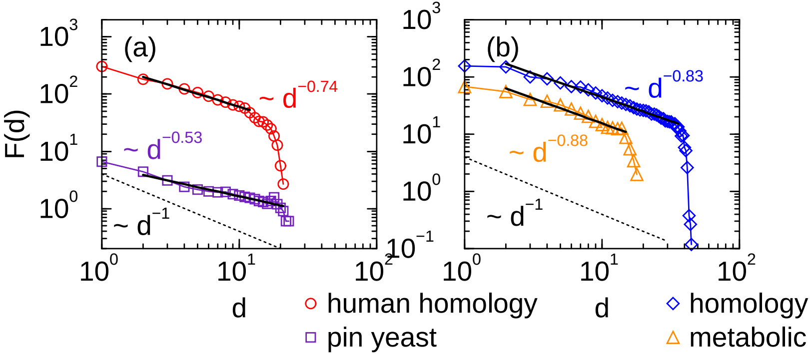
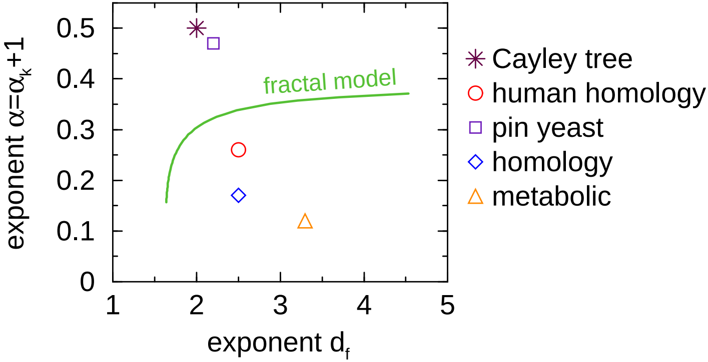


Figure: Degree fluctuation functions for real-world networks.

Comparison with fractal dimension



network	***k*FA**	**box covering** (Song et al.)
human homology	$\alpha_k \simeq -0.74$	$d_f \simeq 2.5$
pin yeast	$\alpha_k \simeq -0.53$	$d_f \simeq 2.2$
homology	$\alpha_k \simeq -0.83$	$d_f \simeq 2.5$
metabolic	$\alpha_k \simeq -0.88$	$d_f \simeq 3.3$

Discussion and outlook

- BA model: exponential decay
- Cayley tree: $\alpha_k = -1/2$ (uncorrelated)
- fractal network model:
  - $e = 0$: $\alpha_k = -1$ (long-range anti-correlated)
  - $e = 1$: exponential decay
- real-world networks: power-law decay
- fluctuation exponent complementary information to fractal dimension

Discussion and outlook

- BA model: exponential decay
- Cayley tree: $\alpha_k = -1/2$ (uncorrelated)
- fractal network model:
  - $e = 0$: $\alpha_k = -1$ (long-range anti-correlated)
  - $e = 1$: exponential decay
- real-world networks: power-law decay
- fluctuation exponent complementary information to fractal dimension

- vary parameters m and x
- analytical description
- also other spatial correlations:
  - various network properties (clustering, betweenness, ...)
  - additional information available (time of addition, activity, ...)

Thank you for your attention!

Manuscript submitted to EPL ...

http://www.rybski.de/diego/