Evidence for power-law anti-correlations in complex networks

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Motivation

Networks consist of simple elements but can comprise complex properties

- broad degree distribution
- clustering
- modularity
- ...
Motivation

Degree correlations

- likelihood that nodes of given degree are connected
- assortative/disassortative mixing: positive/negative correlations
- measures:
  - Pearson correlation coefficient
  - average nearest neighbor degree
  - conditional probability $p(k_1, k_2)$
  - ...

Disassortativity tightly related to *fractality* of complex networks
FIG. 2. The average connectivity $\langle k_{nn} \rangle$ of the nearest neighbors of a node depending on its connectivity $k$ for the 1998 snapshot of the Internet, the generalized BA model with $\gamma = 2.2$ (Ref. [8]), and the fitness model (Ref. [18]). The solid line has a slope $-0.5$. The scattered results for very large $k$ are due to statistical fluctuations.

Figure: R. Pastor-Satorras et al. PRL 2001.
Conditional probability $p(k_1, k_2)$

FIG. 1 (color online). The joint degree distribution $P(k_1, k_2)$ of WWW (top row) and Internet at the router level (bottom row) before renormalization (left), after renormalization forbidding multiple links (center), and including multiple links (right).

Figure: L.K. Gallos et al. PRL 2008.
Motivation

But:
- only correlations between nearest neighbor nodes, i.e. distance 1
- much of the rich topological information gets lost
- how can correlations be measured at larger distances?
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2. Extract the sequence of degrees ($k_i$) along this path of length $d_{ij}$. 
We propose a Degree Fluctuation Analysis ($kFA$):

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2. Extract the sequence of degrees ($k_l$) along this path of length $d_{ij}$.
3. Calculate the average $K_{ij} = \langle k_l \rangle$ of the sequence ($k_l$).
Fluctuation Analysis

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4. Find the shortest paths between all pairs of nodes and determine the corresponding averages, $K_{ij} \ \forall \ i \neq j$. 
We propose a Degree Fluctuation Analysis (kFA):

1. Consider the shortest path between the nodes $i$ and $j$ (if it is not unique, we consider an arbitrary one).
2. Extract the sequence of degrees $(k_l)$ along this path of length $d_{ij}$.
3. Calculate the average $K_{ij} = \langle k_l \rangle$ of the sequence $(k_l)$.
4. Find the shortest paths between all pairs of nodes and determine the corresponding averages, $K_{ij}$ for $i \neq j$.
5. Calculate the fluctuation function $F(d) = \sigma(K_{ij} | d)$, the conditional standard deviation of the $K_{ij}$ at distance $d$.

In analogy to Fluctuation Analysis in time series analysis
If the covariance, $C(d) \sim \langle (k_i - \langle k \rangle)(k_j - \langle k \rangle) | d \rangle$, between the degrees at distance $d$ scales as

- $C(d) \sim d^{-\gamma}$ for positive correlations (assortative) or
- $C(d) \sim -(d^{-\gamma})$ for negative correlations (disassortative)

then we expect

$$F(d) \sim d^{\alpha_k},$$

where $\alpha_k = -\gamma/2$.

Fluctuation exponent differs by 1 from usual Hurst-like exponent: $\alpha = \alpha_k + 1$. 
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Fluctuation exponent differs by 1 from usual Hurst-like exponent: \( \alpha = \alpha_k + 1 \).

\[
\begin{align*}
\alpha_k > -1/2 & \quad \text{positive correlations (assortative)} \\
\alpha_k = -1/2 & \quad \text{uncorrelated} \\
\alpha_k < -1/2 & \quad \text{negative correlations (disassortative)}
\end{align*}
\]
Long-range (anti-) correlations

Figure: Degrees of nodes versus distance along shortest paths for (a) fractal network model and (b) pin yeast network.
1. Barabási-Albert model

Figure: Degree fluctuation functions for the BA model. (a-c) show $F(d)$, (d) shows for $m = 2$ the slopes of exponential fits as a function of the network sizes. 100 configurations.
2. Cayley tree at percolation transition

Figure: $z = 3$ (source: wikipedia)

percolation transition: \[ p_c = \frac{1}{z-1} \]

topological dimension of giant component: \[ d_f = 2 \]
2. Cayley tree at percolation transition

Figure: Degree fluctuation function of the Cayley tree at percolation transition ($z = 3, n = 150$). Dotted maroon lines: quantiles enclosing 90% (100 configurations).
3. Fractal network model


Fractal network:

\[ d_f = \frac{\ln(2m+x)}{\ln(3-2e)} \]

Degree distribution:

\[ p(k) \sim k^{-\left(1 + \frac{\ln(2m+x)}{\ln m}\right)} \]

Figure: generation \( n, m \) new nodes, \( x \) new links, probability \( e \) (source: H.D. Rozenfeld and H.A. Makse, 2009)
3. Fractal network model

Figure: Degree fluctuation functions for the fractal model \((n = 4)\). Dotted maroon lines: quantiles enclosing 90\% (250 configurations). Inset in (c): \(n = 3\) and \(n = 5\) (25 configurations).
3. Fractal network model

Figure: Exponents $\alpha = \alpha_k + 1$ of the fractal network model. Power-law fit for $n = 5$: $\alpha \sim e^{\epsilon}$, $\epsilon \approx 0.2$. 
Figure: Degree fluctuation functions for real-world networks.
Comparison with fractal dimension

**network**
- **human homology**
- **pin yeast**
- **homology**
- **metabolic**

**kFA**
- $\alpha_k \approx -0.74$
- $\alpha_k \approx -0.53$
- $\alpha_k \approx -0.83$
- $\alpha_k \approx -0.88$

**box covering** (Song et al.)
- $d_f \approx 2.5$
- $d_f \approx 2.2$
- $d_f \approx 2.5$
- $d_f \approx 3.3$
Discussion and outlook

- BA model: exponential decay
- Cayley tree: $\alpha_k = -1/2$ (uncorrelated)
- Fractal network model:
  - $\epsilon = 0$: $\alpha_k = -1$ (long-range anti-correlated)
  - $\epsilon = 1$: exponential decay
- Real-world networks: power-law decay
- Fluctuation exponent complementary information to fractal dimension
Discussion and outlook

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- Cayley tree: $\alpha_k = -1/2$ (uncorrelated)
- Fractal network model:
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- Vary parameters $m$ and $x$
- Analytical description
- Also other spatial correlations:
  - Various network properties (clustering, betweenness, ...)
  - Additional information available (time of addition, activity, ...)


Thank you for your attention!

Manuscript submitted to EPL . . .

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